On Certain Class of Harmonic Univalent Functions

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Abstract - A complex-valued functions that are univalent and sense preserving in the unit disk can be written in the form \( f(z) = h(z) + g(z) \), where \( U(z) \) and \( g(z) \) are analytic in \( U \). We will introduced the operator \( D^n \) which defined by convolution involving the polylogarithms functions. Using this operator, we introduce the class \( HP(\alpha, \gamma, n) \) by generalized derivative operator of harmonic univalent functions. We give sufficient coefficient conditions for normalized harmonic functions in the class \( HP(\alpha, \gamma, n) \). These conditions are also shown to be necessary when the coefficients are negative. This leads to distortion bounds and extreme points.

Keywords: Univalent functions, Harmonic functions, Convex combinations, Distortion bounds.

I. INTRODUCTION

Let \( U \) denote the open unit disk and \( S_H \) denote the class of all complex valued harmonic, sense preserving univalent functions \( f(z) \) in \( U \) normalized by \( f(0) = 0 \), \( f_j(0) = 1 \). Each \( f(z) \in S_H \) can be expressed as

\[
 f(z) = h(z) + g(z)
\]  

(1.1)

where \( h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=2}^{\infty} b_k z^k, \quad |b_k| < 1 \)

are analytic in \( U \). A necessary and sufficient condition for \( f(z) \) to be locally univalent. And sense preserving in \( U \) is that

\[
 |h'(z)| > |g'(z)| \quad \text{in} \quad U.
\]

Clunie and Sheil- Small (1984) studied \( S_H \) together with some geometric subclasses of \( S_H \). A function of the form (1.1) is harmonic starlike for \( 1 \leq |z| < r \) if

\[
 \frac{\partial}{\partial \theta} \left( \arg(f(re^{i\theta})) \right) = \Re \left( \frac{zh'(z) - zg'(z)}{h(z) + g(z)} \right) > 0.
\]

Silverman (1998), proved that the coefficient conditions

\[
 \sum_{k=2}^{\infty} k |a_k + b_k| \leq 2 - \alpha, \quad 0 \leq \alpha < 1,
\]

are necessary and sufficient conditions for functions \( f(z) = h(z) + g(z) \) to be harmonic starlike with negative coefficient and harmonic convex with negative coefficient respectively. O. P. Ahuja and J. M. Jahangiri(2005) and S. Yalcin (2005) studied \( S_H \) together with some geometric subclasses of \( S_H \) for \( f(z) = h(z) + g(z) \) given by (1.1).

Definition 1.1 We introduce a differential operator defines as follows: \( D_n f(z) : A \rightarrow A \) by

\[
 D^n f(z) = D (D^{-1} f(z))
\]

(1.2)

where

\[
 D^n h(z) = z + \sum_{k=2}^{\infty} \left[ \frac{1}{2} (k+1) \right]^n a_k z^k
\]

\[
 D^n g(z) = \sum_{k=2}^{\infty} \left[ \frac{1}{2} (k+1) \right]^n b_k z^k, \quad |b_k| < 1
\]

Let \( HP(\alpha, \gamma, n) \) denote the family of harmonic functions \( f \) of the form (1.1) such that

\[
 \Re \left[ \frac{D^n f(z) + \gamma [D^{n+1} f(z) - D^n f(z) \big]}{z} \right] \geq \alpha
\]

where

\[
 \gamma > \frac{1}{2}, \quad 0 \leq \alpha < 1, \quad D^n f(z) \text{ is defined by (1.2)}.
\]

If the co-analytic part of \( f(z) = h(z) + g(z) \) is identically zero, \( n = 0 \) then the family

\( HP(\alpha, \gamma, n) \) turns out to be the class \( F_\alpha \) introduced by Bhoosnurmath and Swamy (1985) for the analytic case.

We further denote by \( HP'(\alpha, \gamma, n) \) the subclass of \( HP(\alpha, \gamma, n) \) such that the functions \( h(z) \) and \( g(z) \) in

\[
 f(z) = h(z) + g(z)
\]

are of the form

\[
 h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = (-1)^n \sum_{k=2}^{\infty} b_k z^k, \quad |b_k| < 1
\]

(1.3)

It is clear that the class \( HP'(\alpha, \gamma, n) \) includes a variety of well-known subclasses of \( S_H \).
In this chapter, we will give the sufficient condition for functions \( f(z) = h(z) + g(z) \) where \( h(z) \) and \( g(z) \) given by (1.1) to be in the class \( HP(\alpha, \gamma, n) \) and it is shown that the coefficient condition is also necessary for the functions in the class \( HP'(\alpha, \gamma, n) \). Coefficient bounds, extreme points, convolution conditions, convex combination of this class are obtained.

II. COEFFICIENT BOUNDS

Firstly, we introduce coefficient contained for function \( HP(\alpha, \gamma, n) \)

**Theorem 2.1** Let \( f(z) = h(z) + g(z) \) be given by (1.1). Furthermore, let

\[
\sum_{k=1}^{\infty} \left[ \frac{1}{2} (k+1) \right]^{\alpha} + \left( 1 - \gamma \right) \left[ \frac{1}{2} (k+1) \right]^{\gamma} \right] |a_k|
\]

where \( a_1 = 1, \gamma > \frac{1}{2}, 0 \leq \alpha < 1 \).

Then \( f \) is harmonic univalent, sense preserving in \( U \) and \( f \in SHP(\alpha, \beta, n, k) \).

**Proof.** Let \( f \in SH(\alpha, \beta, n, k) \). For \( z_1, z_2 \in U \) such that \( z_1 \neq z_2 \), our aim is to prove that \( |f(z_1) - f(z_2)| > 0 \)

\[
|f(z_1) - f(z_2)| = |h(z_1) - h(z_2)| = \left| \sum_{k=1}^{\infty} \frac{1}{2} (k+1) \right| |a_k| |z_1^k - z_2^k|
\]

Hence, \( f \) is univalent in \( U \), \( f(z) \) is sense preserving in \( U \).

This is because

\[
|f'(z)| \geq 1 - \sum_{k=1}^{\infty} |a_k| |z^k| \geq 0
\]

Now, we show that \( f \in HP(\alpha, \gamma, n) \). Using the fact that \( \Re w \geq \alpha \) if and only if \( |1 - \alpha + w| \geq |1 + \alpha - w| \), it suffices to show that

\[
\left| (1 - \alpha) z + D' f(z) + \gamma [D^a f(z) - D^b f(z)] \right|
\]

II-4. COEFFICIENT BOUNDS
Proof. The if part follows from Theorem 2.1 upon noting that the functions \( h(z) \) and \( g(z) \) in \( f \in H^p(\alpha, \gamma, n) \) are of the form (1.3), then \( f \in H^p(\alpha, \gamma, n) \). For the only if part, we show that if \( f \in H^p(\alpha, \gamma, n) \), then the condition (2.4) holds. Note that a necessary and sufficient condition for \( f(z) = h(z) + g(z) \) given by (1.3) be in \( f \in H^p(\alpha, \gamma, n) \) is that

\[
\text{Re}\left\{ \frac{D^* f(z) + \gamma [D^{*+1} f(z) - D^* f(z)]}{z} \right\} \geq \alpha
\]

or, equivalently

\[
\text{Re}(2-\alpha)z + \sum_{k=1}^{\infty} \left[ \gamma \left( \frac{1}{2} k^2 + 1 \right)^{n+1} + (1-\gamma) \left( \frac{1}{2} k^2 + 1 \right)^n \right] |a_k|^2 z^k = 0.
\]

The above required condition (2.5) must hold for all values of \( z \) in \( U \). Upon choosing the values of \( z \) on the positive real axis where \( 0 < |z| = r < 1 \), we must have

\[
(1-\alpha) - \sum_{k=1}^{\infty} \left[ \gamma \left( \frac{1}{2} k^2 + 1 \right)^{n+1} + (1-\gamma) \left( \frac{1}{2} k^2 + 1 \right)^n \right] |a_k|^2 r^k \geq 0.
\]

If the condition (2.1) doesn’t hold, then the numerator in (2.6) is negative for \( r \) sufficiently close to 1. Hence there exists in \( z_0 = r_0 \) (0, 1) for which the quotient in (2.1) is negative. This contradicts the required condition for \( f \in H^p(\alpha, \gamma, n) \) and so the proof is complete.

III. DISTORTION BOUNDS AND EXTREME POINT

First we shall obtain distortion bounds for functions in \( f \in H^p(\alpha, \gamma, n) \).

Theorem 3.1 If \( f \in H^p(\alpha, \gamma, n) \). Then for \( |z| = r < 1 \) we have

\[
|f(z)| \leq \left| (1+|h|^2) r + \frac{1}{3^\gamma} (1-2^\gamma (1+\gamma) |h| - \alpha) r^2 \right|
\]

Proof. Let \( f \in H^p(\alpha, \gamma, m, n) \). Taking the absolute value of \( f' \) we obtain

\[
|f(z)| \leq z + \sum_{k=1}^{\infty} a_k z^k + (1-\alpha) \sum_{k=1}^{\infty} b_k z^k
\]

where \( a_k = 1, \gamma > \frac{1}{2}, 0 \leq \alpha < 1 \).

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\[
\geq \left(1 - |h_1| \right) r - \frac{1}{\left[3^{\gamma + 3}(1 - \gamma)\right]} \left( \sum_{i=1}^{\infty} \left[ \left( \frac{3}{2} \right)^{\gamma} + \left( \frac{3}{2} \right)^{\gamma}(1 - \gamma)\right] |a_i| + \left[ \left( \frac{3}{2} \right)^{\gamma} - \left( \frac{3}{2} \right)^{\gamma}(1 - \gamma)\right] |b_i| \right) r^2
\]

The bounds given in Theorem 3.1. For the functions \( f(z) = h(z) + g(z) \) of the form (1.3) also hold for functions of the form (1.1) if the coefficient condition (2.1) is satisfied functions

\[ f_1(z) = z + \left| b_1 \right| z^2 + \left| c_1 \right| (1 - 2\gamma - 1) |b_1| z^2 \]

and

\[ f_2(z) = (1 - \left| b_1 \right|) z^2 - 2^{\gamma - 1 - 2(\gamma - 1)} |b_1| z^2. \]

For \( |b_1| < 1 - \alpha \) show that the bounds given in Theorem 3.1 is sharp.

The following result follows from the left hand inequality in Theorem 3.1.

**IV. COVERING THEOREM**

The following Covering theorem from the second inequality in Theorem 3.1

**Theorem 4.1** If \( f(z) \in HP_\gamma^* (\alpha, \gamma, m, n) \). Then

\[
\left\{ w : |w| < \frac{3^\gamma(2\gamma + 1) - 2^\gamma + \left[ 2^\gamma(2\gamma - 1) - 3^\gamma(2\gamma + 1) \right] \left| b_1 \right|}{3^\gamma(2\gamma + 1)} \right\} \subset f(U).
\]

**Proof.** By using the covering theorem from the second inequality in Theorem 3.1 when \( r \to 1 \) we have

\[
|w| \leq \left| b_1 \right| - \frac{2^\gamma}{3^\gamma(2\gamma + 1)} (1 - 2\gamma - 1) \left| b_1 \right|
\]

\[
= \frac{3^\gamma(2\gamma + 1) - 2^\gamma + \left[ 2^\gamma(2\gamma - 1) - 3^\gamma(2\gamma + 1) \right] \left| b_1 \right|}{3^\gamma(2\gamma + 1)}
\]

so our proof is complete.

Next, we determine the extreme points of the closed convex hulls of \( f \in HP_\gamma^* (\alpha, \gamma, n) \) denoted by \( clcoHP_\gamma^* (\alpha, \gamma, n) \)

**Theorem 4.2** The function \( f_1 \in clcoHP_\gamma^* (\alpha, \gamma, n) \) if and only if \( f_1 \) can be expressed as

\[
f_1(z) = \sum_{i=1}^{\infty} (X_i h_i + Y_i g_i)
\]

where

\[
h_1(z) = z - \frac{1 - \alpha}{\gamma \left[ \frac{1}{2} (k + 1) \right]! + (1 - \gamma) \left[ \frac{1}{2} (k + 1) \right]!} z^k, \ k = 2, 3, \ldots
\]

\[
g(z) = z + \frac{2^\gamma(1 - \alpha)}{(1 - \gamma) \left[ \frac{1}{2} (k + 1) \right]! - \gamma \left[ \frac{1}{2} (k + 1) \right]!} z^k, \ k = 1, 2, 3, \ldots
\]

\[
\sum_{i=1}^{\infty} (X_i + Y_i) = 1, \ X_i \geq 0 \text{ and } Y_i \geq 0
\]

in particular, the extreme points of \( HP_\gamma^* (\alpha, \gamma, m, n) \). are \( \{h_i\} \) and \( \{g_i\} \).

**Proof.** For the functions \( f_n \) of the form (3.1), we have

\[
f_n(z) = \sum_{i=1}^{\infty} (X_i h_i + Y_i g_i)
\]

\[
= \sum_{i=1}^{\infty} (X_i + Y_i) z - \sum_{i=1}^{\infty} \frac{X_i \left[ \frac{1}{2} (k + 1) \right]! + Y_i \left[ \frac{1}{2} (k + 1) \right]!} {X_i z^k
\]

\[
+ \frac{(1 - \alpha)(1 - \gamma) \left[ \frac{1}{2} (k + 1) \right]!} {Y_i z^k}
\]

then by Theorem 2.1

\[
\sum_{i=1}^{\infty} \frac{\left[ \frac{1}{2} (k + 1) \right]! + (1 - \gamma) \left[ \frac{1}{2} (k + 1) \right]!} {X_i z^k}
\]

\[
+ \sum_{i=1}^{\infty} \frac{(1 - \alpha)(1 - \gamma) \left[ \frac{1}{2} (k + 1) \right]!} {Y_i z^k}
\]

so any \( f \in clcoHP_\gamma^* (\alpha, \gamma, m, n) \).

Conversely, suppose that \( f \in clcoHP_\gamma^* (\alpha, \gamma, m, n) \), set

\[
X_i = \frac{\left[ \frac{1}{2} (k + 1) \right]! + (1 - \gamma) \left[ \frac{1}{2} (k + 1) \right]!} {X_i z^k}
\]

\[
Y_i = \frac{(1 - \alpha)(1 - \gamma) \left[ \frac{1}{2} (k + 1) \right]!} {Y_i z^k}
\]

then note by Theorem 2.2.

0 \leq X_i \leq 1 \text{ and } 0 \leq Y_i \leq 1 \text{ (k=1,2,3,..)
and $X_i = 1 - \sum_{k<i} X_k - \sum_{k>i} Y_k$ therefore, $f$ can be written as

$$f_n(z) = z - \sum_{k=1}^{n} a_k z^k + (1-\gamma) \sum_{k=1}^{n} b_k z^k$$

and $X_i = 1 - \sum_{k<i} X_k - \sum_{k>i} Y_k$ therefore, $f$ can be written as

$$f_n(z) = z - \sum_{k=1}^{n} a_k z^k + (1-\gamma) \sum_{k=1}^{n} b_k z^k$$

the convolution of $f_n$ and $F_n$ is given by

$$(f_n \ast F_n)(z) = (f_n(z) \ast F_n(z)) = z - \sum_{k=1}^{n} A_k z^k + (1-\gamma) \sum_{k=1}^{n} B_k z^k$$

Theorem 5.1. For $0 \leq \beta \leq \alpha < 1$, let $f_n \in HP^\beta(\alpha, \gamma, n)$ and

$F_n \in HP^\beta(\beta, \gamma, n)$

then

$$(f_n \ast F_n) \in HP^\beta(\alpha, \gamma, n)$$

Proof. Since $f_n \in HP^\beta(\alpha, \gamma, n)$ and $F_n \in HP^\beta(\beta, \gamma, n)$ the coefficients of $f_n \ast F_n$ must satisfy the required condition given by Theorem 6.2.2. For $F_n \in HP^\beta(\beta, \gamma, n)$ we note that $|A_k| \leq 1$ and $|B_k| \leq 1$ Now, for the convolution function $f_n \ast F_n$, we obtain

$$\sum_{k=1}^{n} \gamma \left[ \left( \frac{1}{2} k + \frac{1}{2} \right) \right]_{\alpha, \gamma}^{b_k} + (1-\gamma) \left[ \left( \frac{1}{2} k + \frac{1}{2} \right) \right]_{\alpha, \gamma}^{a_k}$$

Theorem 5.2. Let the function $f_{nj}(z)$ defined by (5.1) be in the class $HP^\beta(\alpha, \gamma, n)$ for $j = 1, 2, \ldots, m$. Then the function $t_j(z)$ defined by

$$t_j(z) = \sum_{j=1}^{m} c_j f_{nj}(z), \quad (0 \leq c_j \leq 1)$$

is also in the class $HP^\beta(\alpha, \gamma, n)$ where $\sum_{j=1}^{m} c_j = 1$.

Proof. According to the definition of $t$, we can write

$$t(z) = \sum_{j=1}^{m} \left( \sum_{k=1}^{n} c_{j,k} a_k \right) z^k + (1-\gamma) \sum_{k=1}^{n} \left( \sum_{j=1}^{m} c_{j,b_k} \right) z^k$$

further, since $f_{nj}(z)$ are in $HP^\beta(\alpha, \gamma, n)$ for every ($j=1, 2, \ldots, m$). Then by 2.4 we have
\[
\sum_{k=1}^{n} \left[ \frac{1}{2} (k+1)^{\alpha} \right] + (1-\gamma) \left[ \frac{1}{2} (k+1)^{\beta} \right] \sum_{j=1}^{\infty} |a_{k,j}| \\
\sum_{k=1}^{n} \left[ \frac{1}{2} (k+1)^{\alpha} \right] - (1-\gamma) \left[ \frac{1}{2} (k+1)^{\beta} \right] \sum_{j=1}^{\infty} |b_{k,j}| \\
= \sum_{j=1}^{\infty} c_{j} \left[ \gamma \left( \frac{1}{2} (k+1)^{\alpha} \right) + (1-\gamma) \left( \frac{1}{2} (k+1)^{\beta} \right) \right] \sum_{k=1}^{n} |a_{k,j}| \\
\geq \sum_{j=1}^{\infty} c_{j} = 1.
\]

Hence the theorem follows.

**Corollary 5.1** The family \( HP^\prime (\alpha, \gamma, n) \) is closed under convex combination.

**Proof.** Let the functions \( f_{\mu}(z) \) (\( j = 1, 2 \)) defined by (5.1) be in the class \( HP^\prime (\alpha, \gamma, n) \). Then the function \( \varphi(z) \) defined by

\[
\varphi(z) = \mu f_{\mu}(z) + (1-\mu) f_{\mu}(z) \quad (0 \leq \mu \leq 1)
\]

is in the class \( HP^\prime (\alpha, \gamma, m, n) \), by taking \( m = 2 \), \( c1 = \mu \) and \( c2 = (1-\mu) \) in Theorem 5.2, we have the corollary.

**REFERENCES**


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